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Large deviation principle for noninteracting boson random point processes

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Limit theorems, including the large deviation principle, are established for random point fields (*processes*), which describe the position distributions of the ideal boson gas in presence of the Bose–Einstein condensation. We compare these results with those for the case of the *normal* phase, i.e., without the condensate. *© 2010 American Institute of Physics.* [doi:10.1063/1.3304115]

I. INTRODUCTION AND MAIN RESULTS

Fermion and boson random point processes (fields) were studied by many authors, in particular, since they have a deep connection with the quantum statistical mechanics.[16,18,19,5–8] See also Refs. 14 and 12 for overview and for more references. One of the advantages of the random point field (RPF) approach to quantum statistical mechanical models is that it enables to apply probabilistic limit theorems to these models. In Ref. 15, typical limit theorems are given for a certain class of RPFs which include the particular cases of the fermion as well as boson RPFs. In Ref. 17, the random boson point fields, which describe the position distribution of the constituent particles for the ideal gas in the Bose–Einstein condensation (BEC) regime, are constructed for the first time.

The purpose of the present paper is to give the limit theorems, such as the law of the large numbers, the central limit theorem (CLT), and the large deviation principle (LDP) for the RPFs which describe the boson gases in the presence of the BEC. We compare them with the corresponding theorems for the normal phase (i.e., without the BEC). In the latter case a detailed study of the limit theorems, which do not use the RPF formalism, has been done in Refs. 10 and 9. In the last reference the authors consider even interacting quantum gases, but only in the rarefied region insuring the normal phase. These papers motivated the study of the LDP in the Bogoliubov-type models,[2] where BEC plays a key role in description of thermodynamic behavior and the spectrum of excitations.

The study of the RPF for the BEC is an interesting and delicate mathematical problem,[17] see also a recent paper.[4] It is known that the point process corresponding to the boson RPF is a Cox process driven by the square norm of a Gaussian process. In the normal phase the last process is *centered* and the corresponding Cox process is infinitely divisible. In Ref. 4 it is shown that for the case of the BEC (Ref. 17) the boson RPF is still a Cox process, but now driven by the square norm of a *shifted* Gaussian process. The shift is particle density dependent and preserves the property of the infinite divisibility. In particular, this observation makes a contact with the Dynkin isomorphism theorem (known for Gaussian processes) as well as a relation between infinite divisibility and factorization of the boson RPF in the BEC regime. Therefore, our Theorem 1.3 can be considered as the LDP for the Cox process involved in the BEC.

Let $\{G^\beta := \exp(\beta\Delta)\}_{\beta\geq 0}$ be the (self-adjoint) *heat semigroup* generated by the Laplacian acting

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in the Hilbert space $L^2(\mathbb{R}^d)$ with the scalar product $\langle\cdot,\cdot\rangle$. (Throughout the paper, we assume $d > 2$.) For any non-negative *bounded measurable* function $f \geq 0$ with a *compact support* in $\mathbb{R}^d$, the operator

$$W_f^\beta := (G^\beta(1-G^\beta)^{-1})^{1/2}\sqrt{1-e^{-f}}$$

is a bounded and

$$K_f^\beta := W_f^{\beta *} W_f^\beta$$

is a *trace-class* operator on $L^2(\mathbb{R}^d)$, i.e., $K_f^\beta \in \mathfrak{C}_1(L^2(\mathbb{R}^d))$. We also define K^β, which is unbounded operator with the *translation-invariant* kernel,

$$K^\beta(x,y) := \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d} \frac{e^{ip\cdot(x-y)}}{e^{\beta|p|^2}-1}. \tag{1.1}$$

Below we consider the RPF ν_ρ characterized by the generating functional,[1]

$$\int_{Q(\mathbb{R}^d)} e^{-\langle f,\xi\rangle}\nu_\rho(d\xi) = \frac{\exp(-(\rho-\rho_c)\langle\sqrt{1-e^{-f}},(1+K_f^\beta)^{-1}\sqrt{1-e^{-f}}\rangle)}{\mathrm{Det}[1+K_f^\beta]} \tag{1.2}$$

in the BEC regime: $\rho > \rho_c$. Here $Q(\mathbb{R}^d)$ is the space of all point measures on $\mathbb{R}^d$, Det stands for the *Fredholm determinant* and $\langle f,\xi\rangle := \Sigma_j f(x_j)$, if $\xi = \Sigma_j \delta_{x_j} \in Q(\mathbb{R}^d)$. The *critical* density, $\rho_c := \rho_c(\beta)$, can be expressed as

$$\rho_c(\beta) = K^\beta(x,x) = \frac{\zeta(d/2)}{(4\pi\beta)^{d/2}}. \tag{1.3}$$

The RPF ν_ρ was introduced in Ref. 17 to describe the BEC in the ideal (noninteracting) boson gas. For the detailed presentation of these notions, we refer to Ref. 17. (See also Sec. II.)

Below in the present paper, we use the following notations: $\|\cdot\|_p$ for $L^p(\mathbb{R}^d)$ norm and $\|\cdot\|$ for the norm of the space of all bounded operators on $L^2(\mathbb{R}^d)$.

With these notations the main results of the paper can be expressed as follows.

Theorem 1.1: *(Law of large numbers) For $\kappa \to \infty$, the limit*

$$\frac{1}{\kappa^d}\langle f(\cdot/\kappa),\xi\rangle \to \rho\int_{\mathbb{R}^d} f(x)dx$$

holds in $L^2(Q(\mathbb{R}^d),\nu_\rho)$.

Theorem 1.2. *(CLT) The distribution of the random variable,*

$$Z_\kappa := \frac{1}{\kappa^{(d+2)/2}}\frac{\langle f(\cdot/\kappa),\xi\rangle - \kappa^d\rho\int_{\mathbb{R}^d} f(x)dx}{\sqrt{2(\rho-\rho_c)}\|(-\beta\Delta)^{-1/2}f\|},$$

converges to the standard normal distribution,

$$\lim_{\kappa\to\infty}\int_{Q(\mathbb{R}^d)} e^{itZ_\kappa}\nu_\rho(d\xi) = e^{-t^2/2}.$$

Theorem 1.3. *(LDP) There exists a certain (good) rate convex function $I:\mathbb{R}\mapsto[0,+\infty]$, such that the LDP,*

$$\limsup_{\kappa\to\infty}\frac{1}{\kappa^{d-2}}\log\,\nu_\rho\left(\frac{1}{\kappa^d}\langle f(\cdot/\kappa),\xi\rangle \in F\right) \leq -\inf_{s\in F} I(s) \quad \text{for any closed } F\subset\mathbb{R},$$

$$\liminf_{\kappa\to\infty}\frac{1}{\kappa^{d-2}}\log\, \nu_\rho\left(\frac{1}{\kappa^d}\langle f(\cdot/\kappa),\xi\rangle \in G\right) \geq -\inf_{s\in G} I(s) \quad \text{for any open } G\subset\mathbb{R},$$

holds.

Remark 1.4: It is instructive to compare these results with the corresponding results for the boson RPFs, which describe the boson gas in normal phase. For the law of large numbers, the both results are the same. On the other hand for the CLT and the LDP, there are differences in the tuning of power κ, which indicate that the fluctuations of $\kappa^{-d}\langle f(\cdot/\kappa),\xi\rangle$ around its mean value $\rho\int_{\mathbb{R}^d} f(x)dx$ for the case of BEC are larger than that for normal phase. See Sec. V, for the details.

II. PRELIMINARY ARGUMENTS AND GENERAL SETTING

Let R be a locally compact Hausdorff space with countable basis, and λ be a positive Radon measure on R. We suppose that the non-negative (possibly unbounded) self-adjoint operator K in $L^2(R,\lambda)$ satisfies the following condition $\mathbf{K}^*$.

Condition $\mathbf{K}^*$*:*

(i) *(Locally trace class) For every bounded Borel set $\Lambda\subset R$, $K^{1/2}\chi_\Lambda$ is a Hilbert–Schmidt operator, where χ_Λ denotes the multiplication operator corresponding to the indicator function of the set Λ, which is denoted by the same symbol.*

(ii) *The operator $G=K(1+K)^{-1}$ has non-negative integral kernel $G(x,y)$, which satisfies the conditions*

$$G(x,y) > 0 \quad \lambda\otimes\lambda - a.e.\, (x,y)\in R^2,$$

$$\int_R G(x,y)\lambda(dy) \leq 1 \quad \lambda - a.e.\, x\in R.$$

The above conditions are arranged in such a way that one can simultaneously deal with the random point processes $\mu_K^{\det}$ and $\mu_{K,\rho}$, see Refs. 17 and 15. In particular, the operator K has a positive kernel $K(x,y)$, i.e.,

$$K(x,y) > 0 \quad \lambda\otimes\lambda - a.e.\, (x,y)\in R^2.$$

See Ref. 17. The operator $K_\Lambda := (K^{1/2}\chi_\Lambda)^* K^{1/2}\chi_\Lambda$ is a *trace-class* operator. For a bounded measurable function f with compact support, we define the operator

$$K_f := \sqrt{1-e^{-f}}K_\Lambda\sqrt{1-e^{-f}}, \tag{2.1}$$

where $\operatorname{supp} f\subset\Lambda$. Note that K_f is independent of the choice of Λ, which contains $\operatorname{supp} f$. The equality $K_f=W_f^*W_f$ also holds for $W_f=K^{1/2}\sqrt{1-e^{-f}}$.

Let $Q(R)$ be a Polish space of all locally finite non-negative integer-valued Borel measures on R. Recall that the Borel probability measures on $Q(R)$ (i.e., random point processes on R) $\mu_K^{(\det)}$ and $\mu_{K,\rho}$ are introduced in Refs. 15 and 17 for $\rho>0$ by means of generating functionals,

$$\int_{Q(R)} e^{-\langle f,\xi\rangle}\mu_K^{(\det)}(d\xi) = \mathrm{Det}[1+K_f]^{-1} = \mathrm{Det}[1+(1-e^{-f})K_\Lambda]^{-1}, \tag{2.2}$$

$$\begin{aligned}\int_{Q(R)} e^{-\langle f,\xi\rangle}\mu_{K,\rho}(d\xi) &= \exp\{-\rho\langle\sqrt{1-e^{-f}},(1+K_f)^{-1}\sqrt{1-e^{-f}}\rangle\}\\ &= \exp\{-\rho\langle\chi_\Lambda,(1+(1-e^{-f})K_\Lambda)^{-1}(1-e^{-f})\rangle\}\end{aligned} \tag{2.3}$$

for any non-negative continuous function f and any bounded measurable set $\Lambda\supset\operatorname{supp} f$.

It was shown[17] that for $R=\mathbb{R}^d$ the boson random point process for the ideal Boson gas is described in the region of BEC ($\rho > \rho_c$) by *convolution* of two measures: $\nu_\rho := \mu_K^{(\mathrm{det})} * \mu_{K,\rho-\rho_c}$.

Theorem 2.1: *For any non-negative bounded measurable function f on R with compact support* $\operatorname{supp} f \subset \Lambda$ *in a bounded Borel set $\Lambda \subset R$ one has the following equalities:*

$$(1) \quad \int_{Q(R)} e^{i\langle f,\xi\rangle} \mu_{K,\rho}(d\xi) = \exp[-\rho\langle \chi_\Lambda, (1+(1-e^{if})K_\Lambda)^{-1}(1-e^{if})\rangle],$$

$$(2) \quad \int_{Q(R)} e^{\langle f,\xi\rangle} \mu_{K,\rho}(d\xi) = \begin{cases} \exp[\rho\langle \sqrt{e^f-1}, (1-\sqrt{e^f-1}K_\Lambda\sqrt{e^f-1})^{-1}\sqrt{e^f-1}\rangle] < \infty & \text{for} \quad \|\sqrt{e^f-1}K_\Lambda\sqrt{e^f-1}\| < 1 \\ \infty & \text{for} \quad \|\sqrt{e^f-1}K_\Lambda\sqrt{e^f-1}\| \geq 1, \end{cases}$$

$$(3) \quad \int_{Q(R)} e^{i\langle f,\xi\rangle} \mu_K^{(\mathrm{det})}(d\xi) = \mathrm{Det}[1+(1-e^{if})K_\Lambda]^{-1},$$

$$(4) \quad \int_{Q(R)} e^{\langle f,\xi\rangle} \mu_K^{(\mathrm{det})}(d\xi) = \begin{cases} \mathrm{Det}[1-\sqrt{e^f-1}K_\Lambda\sqrt{e^f-1}]^{-1} < \infty & \text{for} \quad \|\sqrt{e^f-1}K_\Lambda\sqrt{e^f-1}\| < 1 \\ \infty & \text{for} \quad \|\sqrt{e^f-1}K_\Lambda\sqrt{e^f-1}\| \geq 1. \end{cases}$$

Proof: Let $f \not\equiv 0$, i.e., the measure $\lambda(\operatorname{supp} f) > 0$. In Ref. 17, pp. 213–214, it was introduced a family of symmetric non-negative functions $\{\sigma_{\Lambda^n}\}_{n\geq 0}$ defined by the equations,

$$\begin{aligned} \exp[-\rho\langle \sqrt{1-e^{-f}}, (1+K_f)^{-1}\sqrt{1-e^{-f}}\rangle] &= \exp[-\rho\langle \chi_\Lambda, (1+(1-e^{-f})K_\Lambda)^{-1}(1-e^{-f})\rangle] \\ &= \exp\left[-\rho\langle \chi_\Lambda, (1+K_\Lambda)^{-1}\chi_\Lambda\rangle + \rho\sum_{l=0}^{\infty} \langle (1+K_\Lambda)^{-1}\chi_\Lambda, e^{-f}(R_\Lambda e^{-f})^l (1+K_\Lambda)^{-1}\chi_\Lambda\rangle\right] \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \sigma_{\Lambda^n}(x_1,\ldots,x_n) e^{-\sum_{k=1}^n f(x_k)} \lambda^{\otimes n}(dx_1\cdots dx_n). \end{aligned} \tag{2.4}$$

Here, $R_\Lambda = K_\Lambda(1+K_\Lambda)^{-1}$ satisfies $\|R_\Lambda\| < 1$ since K_Λ is a bounded non-negative operator. Using $\{\sigma_{\Lambda^n}\}_{n\geq 0}$, the random point processes $\mu_{K,\rho}$ were defined as the probability measure, such that

$$\int_{Q(R)} F(\xi)\mu_{K,\rho}(d\xi) = \sum_{n=0}^{\infty} \int_{\Lambda^n} \sigma_{\Lambda^n}(x_1,\ldots,x_n) F\left(\sum_{j=1}^n \delta_{x_j}\right) \lambda^{\otimes n}(dx_1,\ldots,dx_n) \tag{2.5}$$

holds for any bounded (or non-negative) measurable functional satisfying $F(\xi) = F(\xi_\Lambda)$, where $\xi_\Lambda(A) = \xi(A \cap \Lambda)$.

From this construction, we obtain the first claim (1),

$$\begin{aligned} \int_{Q(R)} e^{i\langle f,\xi\rangle} \mu_{K,\rho}(d\xi) &= \sum_{n=0}^{\infty} \int_{\Lambda^n} \sigma_{\Lambda^n}(x_1,\cdots,x_n) e^{i\sum_{j=1}^n f(x_j)} \lambda^{\otimes n}(dx_1,\cdots,dx_n) \\ &= \exp\left[-\rho\langle \chi_\Lambda, (1+K_\Lambda)^{-1}\chi_\Lambda\rangle + \rho\sum_{l=0}^{\infty} \langle (1+K_\Lambda)^{-1}\chi_\Lambda, e^{if}(R_\Lambda e^{if})^l (1+K_\Lambda)^{-1}\chi_\Lambda\rangle\right] \\ &= \exp[-\rho\langle \chi_\Lambda, (1+(1-e^{if})K_\Lambda)^{-1}(1-e^{if})\rangle]. \end{aligned}$$

If $z \in \mathbb{C}$ satisfies $|z|e^{\|f\|_\infty} \leq 1$, then we get the equality:

$$\sum_{n=0}^{\infty}\int_{\Lambda^n}\sigma_{\Lambda^n}(x_1,\ldots,x_n)z^n e^{\sum_{j=1}^n f(x_j)}\lambda^{\otimes n}(dx_1,\ldots,dx_n)$$

$$=\exp\left[-\rho\langle\chi_\Lambda,(1+K_\Lambda)^{-1}\chi_\Lambda\rangle+\rho\sum_{l=0}^{\infty}z^{l+1}\langle(1+K_\Lambda)^{-1}\chi_\Lambda,e^f(R_\Lambda e^f)^l(1+K_\Lambda)^{-1}\chi_\Lambda\rangle\right]. \quad (2.6)$$

Since all coefficients of the z-power series in the both sides are non-negative, this equality (2.6) also holds for $z=1$ in the sense that either the both sides are finite and equal or they are both diverge to $+\infty$. When they are finite, we obtain

$$\int_{Q(R)}e^{\langle f,\xi\rangle}\mu_{K,\rho}(d\xi)=\exp[\rho\langle\sqrt{e^f-1},(1-\sqrt{e^f-1}K_\Lambda\sqrt{e^f-1})^{-1}\sqrt{e^f-1}\rangle],$$

cf. the proof of Theorem 2.1 in Ref. 17, pp. 213–214. Hence, for the second claim (2) it is sufficient to show that

$$\text{the finite RHS of}(2.6)\Leftrightarrow\|e^{f/2}R_\Lambda e^{f/2}\|<1\Leftrightarrow\|\sqrt{e^f-1}K_\Lambda\sqrt{e^f-1}\|<1. \quad (2.7)$$

Notice that by Proposition 2.3 (ii) (Ref. 17) the condition $\mathbf{K}^*$ (ii) ensures that $R_\Lambda(x,y)>0$ for $\lambda\otimes\lambda$-almost all $(x,y)\in\Lambda^2$. Since R_Λ is a compact symmetric operator, it follows from the variational principle that $\|e^{f/2}R_\Lambda e^{f/2}\|$ is the largest eigenvalue of the operator $e^{f/2}R_\Lambda e^{f/2}$ with eigenfunction $\varphi_0>0$ (λ a.e. on Λ). Hence we have

$$\langle(1+K_\Lambda)^{-1}\chi_\Lambda,e^f(R_\Lambda e^f)^l(1+K_\Lambda)^{-1}\chi_\Lambda\rangle=|\langle\varphi_0,e^{f/2}(1+K_\Lambda)^{-1}\chi_\Lambda\rangle|^2\|e^{f/2}R_\Lambda e^{f/2}\|^l+O(\|e^{f/2}R_\Lambda e^{f/2}\|^l\delta^l)$$

for some $\delta\in(0,1)$. Note that $|\langle\varphi_0,e^{f/2}(1+K_\Lambda)^{-1}\chi_\Lambda\rangle|>0$ because $(1+K_\Lambda)^{-1}\chi_\Lambda>0$ (λ a.e. on Λ) and $\|(1+K_\Lambda)^{-1}\chi_\Lambda\|>0$. Thus, we get the first equivalence in (2.7).

For the second equivalence, it is enough to prove that

$$\|R_\Lambda^{1/2}e^fR_\Lambda^{1/2}\|<1\Leftrightarrow\|K_\Lambda^{1/2}(e^f-1)K_\Lambda^{1/2}\|<1$$

by duality. Let $\|R_\Lambda^{1/2}e^fR_\Lambda^{1/2}\|=\eta<1$. Then $K_\Lambda\geq 0$, $f\geq 0$, $R_\Lambda^{1/2}e^fR_\Lambda^{1/2}\geq 0$, and

$$1-K_\Lambda^{1/2}(e^f-1)K_\Lambda^{1/2}=(1+K_\Lambda)^{1/2}(1-R_\Lambda^{1/2}e^fR_\Lambda^{1/2})(1+K_\Lambda)^{1/2}$$

imply

$$1-K_\Lambda^{1/2}(e^f-1)K_\Lambda^{1/2}\geq(1+K_\Lambda)(1-\eta)\geq 1-\eta.$$

Hence $K_\Lambda^{1/2}(e^f-1)K_\Lambda^{1/2}\leq\eta<1$. On the other hand, if $\|K_\Lambda^{1/2}(e^f-1)K_\Lambda^{1/2}\|=\theta<1$, then

$$1-R_\Lambda^{1/2}e^fR_\Lambda^{1/2}=(1+K_\Lambda)^{-1/2}(1-K_\Lambda^{1/2}(e^f-1)K_\Lambda^{1/2})(1+K_\Lambda)^{-1/2}\geq(1-\theta)(1+K_\Lambda)^{-1}\geq\frac{1-\theta}{1+\|K_\Lambda\|},$$

which yields

$$0\leq R_\Lambda^{1/2}e^fR_\Lambda^{1/2}\leq 1-\frac{1-\theta}{1+\|K_\Lambda\|}<1.$$

This finishes the proof of claims (1) and (2) of the theorem concerning the measure $\mu_{K,\rho}$.

The claims (3) and (4) concerning the measure $\mu_K^{(\mathrm{det})}$ can be shown similarly if one uses, instead of (2.4), the representation

$$\mathrm{Det}[1+K_f]^{-1}=\mathrm{Det}[1+K_\Lambda]^{-1}\mathrm{Det}[1-e^{-f}R_\Lambda]^{-1}$$

$$=\mathrm{Det}[1+K_\Lambda]^{-1}\sum_{n=0}^{\infty}\frac{1}{n!}\int_{R^n}\mathrm{Per}\{R_\Lambda(x_j,x_k)\}_{1\leq j,k\leq n}e^{-\sum_{l=1}^n f(x_l)}\lambda^{\otimes n}(dx_1,\ldots,dx_n),$$

where Det is the *Fredholm determinant* and Per is the *permanent* of the corresponding matrices.[15] ■

III. OPERATORS

Below we deal with the RPFs which describe the position distribution of the ideal boson gas in $\mathbb{R}^d$ $(d>2)$ *above* the critical density $\rho_c=\rho_c(\beta)$, i.e., in the region of the BEC.

To this end, we take $R=\mathbb{R}^d$ and $K^\beta=G^\beta(1-G^\beta)^{-1}$ for the operator K, where $G^\beta=e^{\beta\Delta}$ for G. Here $\beta>0$ is the inverse temperature and Δ denotes the d-dimensional self-adjoint Laplacian in the space $L^2(\mathbb{R}^d)$ equipped with the Lebesgue measure. Then it can be shown that operator K^β satisfies the condition $\mathbf{K}^*$, see Ref. 17.

In the present section, we derive some miscellaneous properties of these operators, which we use for the proofs in Sec. IV. We adopt the following definition of the Fourier transformation:

$$\tilde{h}(p) \coloneqq (\mathcal{F}h)(p) = \int_{\mathbb{R}^d} e^{-ip\cdot x} h(x) \frac{dx}{(2\pi)^{d/2}}$$

for $h \in L^1(\mathbb{R}^d)$ and its extension to $L^2(\mathbb{R}^d)$.

Lemma 3.1: For any compact $\Lambda\subset\mathbb{R}^d$, the operators $(-\Delta)^{-1/2}\chi_\Lambda$ and $(K^\beta)^{1/2}\chi_\Lambda$ are bounded. Therefore,

$$(-\Delta)_\Lambda^{-1} \coloneqq ((-\Delta)^{-1/2}\chi_\Lambda)^*(-\Delta)^{-1/2}\chi_\Lambda,$$

$$K_\Lambda^\beta \coloneqq ((K^\beta)^{1/2}\chi_\Lambda)^*(K^\beta)^{1/2}\chi_\Lambda$$

are bounded non-negative self-adjoint operators.

Proof: These properties can be verified with a help of the Fourier transformation. For any $g \in L^2(\mathbb{R}^d)$, we obtain

$$\begin{aligned}
\|(-\Delta)^{-1/2}\chi_\Lambda g\|^2 &= \int_{\mathbb{R}^d} \frac{|\widetilde{\chi_\Lambda g}(p)|^2}{|p|^2} dp \\
&\le \int_{|p|\le 1} \frac{\|\widetilde{\chi_\Lambda g}\|_\infty^2}{|p|^2} dp + \int_{\mathbb{R}^d} |\widetilde{\chi_\Lambda g}(p)|^2 dp \\
&\le c_1 \|\chi_\Lambda g\|_{L^1}^2 + \|\chi_\Lambda g\|^2 \le c_2 \|\chi_\Lambda\|_2^2 \|g\|^2 + \|\chi_\Lambda\|_\infty^2 \|g\|^2 \\
&= (1 + c|\Lambda|)\|g\|^2.
\end{aligned}$$

Thus, $(-\Delta)^{-1/2}\chi_\Lambda$ is bounded and $\|(-\Delta)^{-1/2}\chi_\Lambda\| \le \sqrt{1+c|\Lambda|}$ holds. It gives $\|(-\Delta)_\Lambda^{-1}\| \le 1+c|\Lambda|$. Here, $|\Lambda|$ denotes the Lebesgue measure of Λ.

A similar argument is valid for the operator K_Λ^β. ■

Definition 3.2: For $\kappa>0$, we define the transformation

$$U_\kappa : L^2(\mathbb{R}^d) \ni g(\cdot) \mapsto \kappa^{d/2} g(\kappa\cdot) \in L^2(\mathbb{R}^d).$$

Lemma 3.3: The transformation U_κ is unitary on $L^2(\mathbb{R}^d)$ for any $\kappa>0$, and it has the following properties.

(1) $U_\kappa h U_\kappa^{-1} = h(\kappa\cdot)$ for the multiplication operator by function h.
(2) $U_\kappa \Delta U_\kappa^{-1} = \kappa^{-2}\Delta$.
(3) $U_\kappa(-\Delta)_{\kappa\Lambda}^{-1}U_\kappa^{-1} = \kappa^2(-\Delta)_\Lambda^{-1}$, $U_\kappa G^\beta U_\kappa^{-1} = G^{\beta/\kappa^2}$.
(4) $U_\kappa K_{\kappa\Lambda}^\beta U_\kappa^{-1} = K_\Lambda^{\beta/\kappa^2}$.

Here we define $\kappa\Lambda \coloneqq \{x \in \mathbb{R}^d : x/\kappa \in \Lambda\}$.

Proof: These properties are a straightforward consequence of the relation $\mathcal{F}U_\kappa = U_\kappa^{-1}\mathcal{F}$ on

$L^2(\mathbb{R}^d)$. ■

Definition 3.4: For bounded non-negative function f with a compact support and for $\kappa > 0$, we put

$$f_\kappa^{(\pm)}(x) := \pm \kappa^2 (e^{\pm f(x)/\kappa^2} - 1).$$

Lemma 3.5: One has the following estimates:

$$f_\kappa^{(\pm)}(x) \geq 0, \quad \left| \chi_{\{f>0\}}(x) \frac{f_\kappa^{(\pm)}(x)}{f(x)} \right| \leq e^{\|f\|_\infty/\kappa^2}, \quad \chi_{\{f>0\}}(x) \left| 1 - \sqrt{\frac{f_\kappa^{(\pm)}(x)}{f(x)}} \right| \leq \frac{\|f\|_\infty}{2\kappa^2} e^{\|f\|_\infty/\kappa^2},$$

$$\|f_\kappa^{(\pm)}\|_\infty \leq \|f\|_\infty e^{\|f\|_\infty/\kappa^2}, \quad \|f - f_\kappa^{(\pm)}\|_\infty \leq \frac{\|f\|_\infty}{2\kappa^2} e^{\|f\|_\infty/\kappa^2}.$$

Proof: These estimates are a direct consequence of the elementary inequalities,

$$\frac{|e^y - 1|}{|y|} \leq e^{|y|}, \quad \frac{|e^y - 1 - y|}{|y|} \leq \frac{|y| e^{|y|}}{2},$$

and $|\sqrt{z} - 1| \leq |z - 1|$ for $y \in \mathbb{R} - \{0\}$, $z \geq 0$. ■

Lemma 3.6: For any $\kappa > 0$, we have the estimates

$$0 \leq (-\beta\Delta)_\Lambda^{-1} - \kappa^{-2} K_\Lambda^{\beta/\kappa^2} \leq (2\kappa^2)^{-1}.$$

Proof: Using the Fourier transformation, we get

$$\langle g, [\kappa^2(-\beta\Delta)_\Lambda^{-1} - K_\Lambda^{\beta/\kappa^2}] g\rangle = \int_{\mathbb{R}^d} \left(\frac{\kappa^2}{\beta|p|^2} - \frac{1}{e^{\beta|p|^2/\kappa^2} - 1} \right) |\widetilde{\chi_\Lambda g}(p)|^2 dp.$$

Then lemma follows from the inequality

$$0 \leq \frac{1}{y} - \frac{1}{e^y - 1} \leq \frac{1}{2} \quad \text{for } y > 0,$$

and from the estimate $\|\widetilde{\chi_\Lambda g}\| = \|\chi_\Lambda g\| \leq \|g\|$. ■

Lemma 3.7: Suppose that supp $f \subset \Lambda$. *Then for $\kappa \to \infty$ one gets the operator-norm asymptotic,*

$$\|\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f} - \kappa^{-2}\sqrt{f_\kappa^{(\pm)}} K_\Lambda^{\beta/\kappa^2} \sqrt{f_\kappa^{(\pm)}}\| = O(\kappa^{-2}),$$

in the space $L^2(\mathbb{R}^d)$.

Proof: From Lemma 3.1, 3.5, and 3.6, we obtain

$$\begin{aligned}
&\|\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f} - \kappa^{-2}\sqrt{f_\kappa^{(\pm)}} K_\Lambda^{\beta/\kappa^2} \sqrt{f_\kappa^{(\pm)}}\| \\
&\quad \leq \|(\sqrt{f} - \sqrt{f_\kappa^{(\pm)}})(-\beta\Delta)_\Lambda^{-1}\sqrt{f}\| + \|\sqrt{f_\kappa^{(\pm)}}(-\beta\Delta)_\Lambda^{-1}(\sqrt{f} - \sqrt{f_\kappa^{(\pm)}})\| \\
&\quad + \|\sqrt{f_\kappa^{(\pm)}}[(-\beta\Delta)_\Lambda^{-1} - \kappa^{-2} K_\Lambda^{\beta/\kappa^2}]\sqrt{f_\kappa^{(\pm)}}\| \\
&\quad \leq (\|\sqrt{f}\|_\infty + \|\sqrt{f_\kappa^{(\pm)}}\|_\infty)\|(-\beta\Delta)_\Lambda^{-1}\|\|\sqrt{f} - \sqrt{f_\kappa^{(\pm)}}\|_\infty \\
&\quad + \|\sqrt{f_\kappa^{(\pm)}}\|_\infty^2 \|(-\beta\Delta)_\Lambda^{-1} - \kappa^{-2} K_\Lambda^{\beta/\kappa^2}\| = O(\kappa^{-2}).
\end{aligned}$$

■

Lemma 3.8: The operator $K_\Lambda^{\beta/\kappa^2}$ is a trace-class operator on $L^2(\mathbb{R}^d)$, i.e., $K_\Lambda^{\beta/\kappa^2} \in \mathfrak{C}_1(L^2(\mathbb{R}^d))$ and

$$\mathrm{Tr}[\sqrt{f}K_\Lambda^{\beta/\kappa^2}\sqrt{f}] = \kappa^d\rho_c\int_{\mathbb{R}^d} f(x)dx. \tag{3.1}$$

Proof: Let $\{\phi_n\}_n$ be a *complete orthonormal system* (CONS) in $L^2(\mathbb{R}^d)$ and let $g(x) = e^{ip\cdot x}\chi_\Lambda(x)$. Then we have

$$\sum_n |\widetilde{\chi_\Lambda\phi_n}(p)|^2 = \sum_n \frac{|\langle g,\phi_n\rangle|^2}{(2\pi)^d} = \frac{\|g\|^2}{(2\pi)^d} = \frac{\|\chi_\Lambda\|^2}{(2\pi)^d}.$$

This yields

$$\sum_n \langle\phi_n, K_\Lambda^{\beta/\kappa^2}\phi_n\rangle = \sum_n\int_{\mathbb{R}^d}\frac{1}{e^{\beta|p|^2/\kappa^2}-1}|\widetilde{\chi_\Lambda\phi_n}(p)|^2dp = \|\chi_\Lambda\|_2^2\int_{\mathbb{R}^d}\frac{1}{e^{\beta|p|^2/\kappa^2}-1}\frac{dp}{(2\pi)^d} = \kappa^d\rho_c|\Lambda| < \infty.$$

Since $K_\Lambda^{\beta/\kappa^2}\geq 0$, it follows that $K_\Lambda^{\beta/\kappa^2}\in\mathfrak{C}_1(L^2(\mathbb{R}^d))$. Similarly, we obtain the explicit value (3.1). ∎

Lemma 3.9: The operator $K_\Lambda^{\beta/\kappa^2}\geq 0$ verifies the following Hilbert–Schmidt norm estimate from above:

$$\|K_\Lambda^{\beta/\kappa^2}\|_{\mathrm{HS}}^2 \leq c_d(\kappa^2/\beta)^{(d\vee 4)/2}(1+|\log(\kappa^2/\beta)|)|\Lambda|(1+|\Lambda|). \tag{3.2}$$

Here c_d is a constant depending only on the dimension $d>2$.

Proof: By the Fourier transformation, we obtain

$$\|K_\Lambda^{\beta/\kappa^2}\|_{\mathrm{HS}}^2 = \int_{\mathbb{R}^d}\frac{dq}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}\frac{dp}{(2\pi)^{d/2}}\frac{|\widetilde{\chi_\Lambda}(p-q)|^2}{(e^{\beta|p|^2/\kappa^2}-1)(e^{\beta|q|^2/\kappa^2}-1)} \tag{3.3}$$

$$= \int_{\mathbb{R}^d}\frac{dq}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}\frac{dp}{(2\pi)^{d/2}}\frac{|\widetilde{\chi_\Lambda}(p)|^2}{(e^{\beta|p+q|^2/\kappa^2}-1)(e^{\beta|q|^2/\kappa^2}-1)}. \tag{3.4}$$

Case: $2<d<4$. From (3.4), we obtain the estimate

$$\begin{aligned}\|K_\Lambda^{\beta/\kappa^2}\|_{\mathrm{HS}}^2 &\leq \int_{\mathbb{R}^d}\frac{dq}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}\frac{dp}{(2\pi)^{d/2}}\frac{\kappa^4|\widetilde{\chi_\Lambda}(p)|^2}{\beta^2|p+q|^2|q|^2}\\ &= \left(\frac{\kappa^2}{\beta}\right)^{4/2}\int_{\mathbb{R}^d}\frac{|\widetilde{\chi_\Lambda}(p)|^2}{|p|^{4-d}}\frac{dp}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{d\tilde{q}}{|e+\tilde{q}|^2|\tilde{q}|^2}\\ &\leq \left(\frac{\kappa^2}{\beta}\right)^{4/2}c\left[\int_{|p|\leq 1}\frac{\|\widetilde{\chi_\Lambda}\|_\infty^2}{|p|^{4-d}}dp + \int_{|p|>1}|\widetilde{\chi_\Lambda}(p)|^2dp\right]\\ &\leq \left(\frac{\kappa^2}{\beta}\right)^{4/2}c(\|\chi_\Lambda\|_{L^1}^2+\|\chi_\Lambda\|^2) = \left(\frac{\kappa^2}{\beta^2}\right)^{4/2}c(|\Lambda|^2+|\Lambda|).\end{aligned}$$

Here we changed the variable $q=|p|\tilde{q}$ in the first equality, where e denotes a unit vector in $\mathbb{R}^d$.

Case: $d>4$. We apply the Cauchy–Schwarz inequality to (3.3) to get

$$\begin{aligned}\|K_\Lambda^{\beta/\kappa^2}\|_{\mathrm{HS}}^2 &\leq \sqrt{\int\int\frac{|\widetilde{\chi_\Lambda}(p-q)|^2dpdq}{(e^{\beta|p|^2/\kappa^2}-1)^2(2\pi)^d}}\sqrt{\int\int\frac{|\widetilde{\chi_\Lambda}(p-q)|^2dpdq}{(e^{\beta|q|^2/\kappa^2}-1)^2(2\pi)^d}}\\ &= \int_{\mathbb{R}^d}|\widetilde{\chi_\Lambda}(p)|^2\frac{dp}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{dq}{(e^{\beta|q|^2/\kappa^2}-1)^2}\\ &= c_d|\Lambda|\left(\frac{\kappa^2}{\beta}\right)^{d/2}.\end{aligned}$$

Case: $d=4$. Let us decompose (3.4) into two parts,

$$\|K_\Lambda^{\beta/\kappa^2}\|_{\mathrm{HS}}^2=\int_{\mathbb{R}^d}\frac{dp}{(2\pi)^d}|\widetilde{\chi_\Lambda}(p)|^2\left\{\int_{|q|\geq 2|p|}+\int_{|q|<2|p|}\right\}\frac{dq}{(e^{\beta|p+q|^2/\kappa^2}-1)(e^{\beta|q|^2/\kappa^2}-1)}=\mathcal{I}_1+\mathcal{I}_2.$$

For $\mathcal{I}_1$, $|q|\geq 2|p|$ implies $|p+q|\geq|q|-|p|\geq|q|/2$. Therefore, it follows that

$$\begin{aligned}\mathcal{I}_1&\leq\int_{\mathbb{R}^d}\frac{dp}{(2\pi)^d}|\widetilde{\chi_\Lambda}(p)|^2\int_{|q|\geq 2|p|}\frac{dq}{(e^{\beta|q|^2/4\kappa^2}-1)(e^{\beta|q|^2/\kappa^2}-1)}\\
&\leq\int_{\mathbb{R}^d}\frac{dp}{(2\pi)^d}|\widetilde{\chi_\Lambda}(p)|^2\left\{\theta(1-2|p|)\int_{1\geq|q|\geq 2|p|}+\int_{\kappa/\sqrt{\beta}\geq|q|>1}+\int_{|q|>\kappa/\sqrt{\beta}}\right\}\\
&\quad\times\frac{dq}{(e^{\beta|q|^2/4\kappa^2}-1)(e^{\beta|q|^2/\kappa^2}-1)}\\
&\leq\int_{\mathbb{R}^d}\frac{dp}{(2\pi)^d}|\widetilde{\chi_\Lambda}(p)|^2\left\{\theta(1-2|p|)\int_{1\geq|q|\geq 2|p|}\frac{4\kappa^4dq}{\beta^2|q|^4}+\int_{\kappa/\sqrt{\beta}\geq|q|>1}\frac{4\kappa^4dq}{\beta^2|q|^4}\right.\\
&\quad\left.+\int_{|q|>\kappa/\sqrt{\beta}}\frac{dq}{(e^{\beta|q|^2/4\kappa^2}-1)(e^{\beta|q|^2/\kappa^2}-1)}\right\}\\
&\leq\int_{\mathbb{R}^d}\frac{dp}{(2\pi)^d}|\widetilde{\chi_\Lambda}(p)|^2\left\{\theta(1-2|p|)\frac{\kappa^4}{\beta^2}c_1\log\frac{1}{2|p|}+\frac{\kappa^4}{\beta^2}c_2\left|\log\left(\frac{\kappa^2}{\beta}\right)\right|\right.\\
&\quad\left.+\left(\frac{\kappa^2}{\beta}\right)^2\int_{|\tilde{q}|>1}\frac{d\tilde{q}}{(e^{|\tilde{q}|^2/4}-1)(e^{|\tilde{q}|^2}-1)}\right\}\\
&\leq\|\widetilde{\chi_\Lambda}\|_\infty^2\left(\frac{\kappa^2}{\beta}\right)^{4/2}c_1\int_{|p|\leq 1/2}\log\frac{1}{2|p|}dp+\left(\frac{\kappa^2}{\beta}\right)^{4/2}\left(c_2\left|\log\left(\frac{\kappa^2}{\beta}\right)\right|+c_3\right)\|\chi_\Lambda\|_2^2\\
&\leq c_4\left(\frac{\kappa^2}{\beta}\right)^{4/2}(1+|\log(\kappa^2/\beta)|)(|\Lambda|+|\Lambda|^2).\end{aligned}$$

For $\mathcal{I}_2$, we obtain:

$$\begin{aligned}\mathcal{I}_2&\leq\int_{\mathbb{R}^4}\frac{dp}{(2\pi)^d}|\widetilde{\chi_\Lambda}(p)|^2\int_{|q|<2|p|}\frac{\kappa^4dq}{\beta^2|p+q|^2|q|^2}\\
&=\int_{\mathbb{R}^d}\frac{dp}{(2\pi)^d}|\widetilde{\chi_\Lambda}(p)|^2\int_{|\tilde{q}|<2}\frac{\kappa^4d\tilde{q}}{\beta^2|e+\tilde{q}|^2|\tilde{q}|^2}\\
&=c\left(\frac{\kappa^2}{\beta}\right)^{4/2}|\Lambda|.\end{aligned}$$

Thus, we have obtained the desired estimate (3.2) for all cases. ■

IV. LIMIT THEOREMS IN CONDENSATE PHASE

In this section, we consider the limit theorems for the RPF described in domain with the BEC: $\rho>\rho_c=K^\beta(x,x)$, by the convolution of measures: $\nu_\rho=\mu_{K^\beta}^{(\mathrm{det})}*\mu_{K^\beta,\rho-\rho_c}$, see Theorem 2.1.

Recall that for any f there exists $\Lambda\subset\mathbb{R}^d$, such that supp $f\subset\Lambda$, and we use this everywhere below.

Proposition 4.1: For a bounded measurable set $\Lambda\subset\mathbb{R}^d$ *and non-negative bounded function* f *with* supp $f\subset\Lambda$*, one gets the equalities,*

$$\int_{Q(\mathbb{R}^d)} \langle f,\xi\rangle \nu_\rho(d\xi) = \rho \int_{\mathbb{R}^d} f(x)dx$$

and

$$\int_{Q(\mathbb{R}^d)} \left(\langle f,\xi\rangle - \int_{Q(\mathbb{R}^d)} \langle f,\xi\rangle \nu_\rho(d\xi) \right)^2 \nu_\rho(d\xi) = \rho \int_{\mathbb{R}^d} f(x)^2 dx + \mathrm{Tr}[fK_\Lambda^\beta f K_\Lambda^\beta] + 2(\rho-\rho_c)\langle f, K_\Lambda^\beta f\rangle .$$

Proof: Let us put

$$e^{-W(f)} := \int_{Q(\mathbb{R}^d)} e^{-\langle f,\xi\rangle} \nu_\rho(d\xi).$$

Then we get

$$W(f) = (\rho-\rho_c)\langle \chi_\Lambda, (1+(1-e^{-f})K_\Lambda^\beta)^{-1}(1-e^{-f})\rangle + \log \mathrm{Det}[1+(1-e^{-f})K_\Lambda^\beta]$$

from (2.2) and (2.3). For small $\epsilon > 0$, this yields the expansion

$$W(\epsilon f) = \epsilon\rho \int_{\mathbb{R}^d} f(x)dx - \frac{\epsilon^2}{2}\rho\int_{\mathbb{R}^d} f(x)^2 dx - \frac{\epsilon^2}{2}\mathrm{Tr}[fK_\Lambda^\beta f K_\Lambda^\beta] - \epsilon^2(\rho-\rho_c)\langle f, K_\Lambda^\beta f\rangle + O(\epsilon^3),$$

which implies the proposition. ■

Corollary 4.2: Under the same conditions as in the Proposition 4.1, one obtains, for large κ, the following asymptotic behavior:

$$\int_{Q(\mathbb{R}^d)} \langle f(./\kappa),\xi\rangle \nu_\rho(d\xi) = \kappa^d \rho \int_{\mathbb{R}^d} f(x)dx + o(\kappa^d)$$

and

$$\int_{Q(\mathbb{R}^d)} \left(\langle f(./\kappa),\xi\rangle - \int_{Q(\mathbb{R}^d)} \langle f(./\kappa),\xi\rangle \nu_\rho(d\xi) \right)^2 \nu_\rho(d\xi) = 2\kappa^{d+2}(\rho-\rho_c)\langle f, (-\beta\Delta)_\Lambda^{-1} f\rangle + O(\kappa^{4\vee d}\log\kappa).$$

Proof: Using the unitary operator U_κ, we get

$$\begin{aligned}\mathrm{Tr}[f(\cdot/\kappa)K_{\kappa\Lambda}^\beta f(\cdot/\kappa)K_{\kappa\Lambda}^\beta] &= \mathrm{Tr}[U_\kappa f(\cdot/\kappa)K_{\kappa\Lambda}^\beta f(\cdot/\kappa)K_{\kappa\Lambda}^\beta U_\kappa^{-1}] = \mathrm{Tr}[fK_\Lambda^{\beta/\kappa^2} f K_\Lambda^{\beta/\kappa^2}] \le \|f\|_\infty^2 \|K_\Lambda^{\beta/\kappa^2}\|_{\mathrm{HS}}^2 \\ &= O(\kappa^{d\vee 4}\log\kappa)\end{aligned}$$

and

$$\langle f(\cdot/\kappa), K_{\kappa\Lambda}^\beta f(\cdot/\kappa)\rangle = \langle U_\kappa f(\cdot/\kappa), U_\kappa K_{\kappa\Lambda}^\beta f(\cdot/\kappa)\rangle = \kappa^d \langle f, K_\Lambda^{\beta\kappa^{-2}} f\rangle = \kappa^{d+2}\langle f, (-\beta\Delta)_\Lambda^{-1} f\rangle + O(\kappa^d).$$

Here we used Lemma 3.9 and Lemma 3.6. Since $\mathrm{supp}\, f(\cdot/\kappa) \subset \kappa\Lambda = \{x\in\mathbb{R}^d : x/\kappa \in \Lambda\}$, then Proposition 4.1 yields the corollary. ■

Theorem 4.3: *(Law of large numbers) For $\kappa\to\infty$ and for any bounded function f with compact support the limit*

$$\frac{1}{\kappa^d}\langle f(./\kappa),\xi\rangle \to \rho\int_{\mathbb{R}^d} f(x)dx$$

holds in $L^2(Q(\mathbb{R}^d),\nu_\rho)$.

Proof: This is a simple consequence of the Corollary 4.2. ■

Remark 4.4: If κ tends to infinity taking its values in $\mathbb{N}$, then the above theorem holds almost *sure. This can be shown by the standard argument using the Tchebyshev inequality and the first Borel–Cantelli lemma.*

Theorem 4.5: *(CLT) For $\kappa \to \infty$, the family of random variables,*

$$Z_\kappa = \frac{1}{\kappa^{(d+2)/2}} \frac{\langle f(\cdot/\kappa),\xi\rangle - \rho\kappa^d \int_{\mathbb{R}^d} f(x)dx}{\sqrt{2(\rho-\rho_c)\langle f,(-\beta\Delta)^{-1}f\rangle}},$$

converges in distribution to the standard Gaussian random variable.

Proof: Since supp $f \subset \Lambda$, by Theorem 2.1, parts (1) and (3), we obtain

$$\mathbb{E}_{\nu_\rho}\left[\exp\left[i\lambda\kappa^{-(d+2)/2}\left(\langle f(\cdot/\kappa),\xi\rangle - \rho\kappa^d\int_{\mathbb{R}^d} f(x)dx\right)\right]\right] = \exp\left[-i\lambda\kappa^{(d-2)/2}\rho\int_{\mathbb{R}^d} f(x)dx - W_\kappa\right],$$

where

$$W_\kappa = (\rho-\rho_c)\langle \chi_{\kappa\Lambda},(1+(1-e^{i\lambda\kappa^{-(d+2)/2}f(\cdot/\kappa)})K^\beta_{\kappa\Lambda})^{-1}(1-e^{i\lambda\kappa^{-(d+2)/2}f(\cdot/\kappa)})\rangle$$
$$+\log \mathrm{Det}[1+(1-e^{i\lambda\kappa^{-(d+2)/2}f(\cdot/\kappa)})K^\beta_{\kappa\Lambda}].$$

By definition of transformation U_κ and by Lemma 3.6, the first term can be expanded as

$$(\rho-\rho_c)\langle U_\kappa\chi_{\kappa\Lambda},U_\kappa(1+(1-e^{i\lambda\kappa^{-(d+2)/2}f(\cdot/\kappa)})K^\beta_{\kappa\Lambda})^{-1}(1-e^{i\lambda\kappa^{-(d+2)/2}f(\cdot/\kappa)})\rangle$$
$$= -i\lambda(\rho-\rho_c)\kappa^{(d-2)/2}\left[\int_{\mathbb{R}^d} f dx + i\lambda\kappa^{-(d+2)/2}\langle f,K_\Lambda^{\beta/\kappa^2}f\rangle\right] + o(1)$$
$$= -i\lambda(\rho-\rho_c)\kappa^{(d-2)/2}\int_{\mathbb{R}^d} f dx + \lambda^2(\rho-\rho_c)\langle f,(-\beta\Delta)_\Lambda^{-1}f\rangle + o(1).$$

Here we applied the bound,

$$\|(1-Y)^{-1}-(1+Y)\| \le c\|Y\|^2,$$

which is valid for operators with small enough operator norms.

Similarly, we also get the representation for the second term,

$$\log \mathrm{Det}[1+(1-e^{i\lambda\kappa^{-(d+2)/2}f})K_\Lambda^{\beta/\kappa^2}] = -i\lambda\kappa^{-(d+2)/2}\,\mathrm{Tr}[fK_\Lambda^{\beta/\kappa^2}] + R,$$

where

$$\mathrm{Tr}[fK_\Lambda^{\beta/\kappa^2}] = \rho_c\kappa^d\int_{\mathbb{R}^d} f(x)dx$$

and

$$|R| \le \mathrm{Tr}[((1-e^{i\lambda\kappa^{-(d+2)/2}f})K_\Lambda^{\beta/\kappa^2})^2] = O(\lambda^2\kappa^{-d-2})\|f\|_\infty^2\|K_\Lambda^{\beta/\kappa^2}\|_{\mathrm{HS}}^2 = o(1).$$

Here we used the bound,

$$|\log \mathrm{Det}[1+Y] - \mathrm{Tr}\,Y| = |\log \mathrm{Det}_2[1+Y]| = O(\|Y\|_{\mathrm{HS}}^2) \tag{4.1}$$

for the trace-class operators with small operator norms. Recall that $\mathrm{Det}_2[1+Y] := e^{-\mathrm{Tr}\,Y}\,\mathrm{Det}[1+Y] = \mathrm{Det}[(1+Y)e^{-Y}]$ denotes a "regularized" determinant, which can be extended to the Hilbert–Schmidt operators Y, see, e.g., Ref. 15.

Thus we get

$$W_\kappa = -i\lambda\kappa^{(d-2)/2}\rho\int f dx + \lambda^2(\rho-\rho_c)\langle f,(-\beta\Delta)_\Lambda^{-1}f\rangle + o(1),$$

and since $\langle f,(-\beta\Delta)_\Lambda^{-1}f\rangle = \langle f,(-\beta\Delta)^{-1}f\rangle$, see Lemma 3.1, one has

$$\mathbb{E}_{\nu_\rho}\left[\exp\left[i\lambda\kappa^{-(d+2)/2}\left(\langle f(\cdot/\kappa),\xi\rangle - \rho\kappa^d\int_{\mathbb{R}^d} f(x)dx\right)\right]\right] = e^{-\lambda^2(\rho-\rho_c)\langle f,(-\beta\Delta)^{-1}f\rangle + o(1)}.$$

Then setting $\lambda := t/\sqrt{2(\rho-\rho_c)\langle f,(-\beta\Delta)^{-1}f\rangle}$, we finally obtain the limit,

$$\mathbb{E}_{\nu_\rho}[e^{itZ_\kappa}] \to e^{-t^2/2},$$

which finishes the proof of the CLT. ■

Remark 4.6: The above calculations show that the value of the variation, which we need to normalize the limit of random variables Z_κ, is contributed from the measure $\mu_{K^\beta,\rho-\rho_c}$, see Theorem 2.1, part (1) and the convolution $\nu_\rho = \mu_{K^\beta}^{(\mathrm{det})} * \mu_{K^\beta,\rho-\rho_c}$.

Before we pass to the *LDP*, we prove the following lemma.

Lemma 4.7: Let $\|\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}\| < 1$. Then $-\beta\Delta - f$ is a self-adjoint operator, which satisfies the property: $\mathrm{Spec}(-\beta\Delta - f) \subset [0,\infty)$. *Moreover, the operator $(-\beta\Delta - f)_\Lambda^{-1}$ is bounded and we have*

$$\langle\sqrt{f},[1-\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}]^{-1}\sqrt{f}\rangle = \int_{\mathbb{R}^d} f(x)dx + \langle f,(-\beta\Delta - f)_\Lambda^{-1}f\rangle. \tag{4.2}$$

Proof: Since the operator $-\beta\Delta$ is self-adjoint, the spectrum $\mathrm{Spec}(-\beta\Delta) \subset [0,\infty)$, and f is a bounded function, it is obvious that $-\beta\Delta - f$ is self-adjoint and $(\delta - \beta\Delta)^{-1}$ is bounded non-negative operator for arbitrary $\delta > 0$. Since $f \geq 0$ and supp $f \subset \Lambda$, it is also obvious that

$$0 \leq \sqrt{f}(\delta - \beta\Delta)^{-1}\sqrt{f} \leq \sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}.$$

Together with the assumption $\|\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}\| < 1$, the operator

$$S := (\delta - \beta\Delta)^{-1} + (\delta - \beta\Delta)^{-1}\sqrt{f}\sum_{n=0}^{\infty}(\sqrt{f}(\delta - \beta\Delta)^{-1}\sqrt{f})^n\sqrt{f}(\delta - \beta\Delta)^{-1} \tag{4.3}$$

is a bounded non-negative operator. On the other hand, one can check that

$$(\delta - \beta\Delta - f)S = I \quad \text{and} \quad S(\delta - \beta\Delta - f) = I_{\mathrm{Dom}(\Delta)},$$

which implies that $S = (\delta - \beta\Delta - f)^{-1}$. Thus, we have $-\delta \notin \mathrm{Spec}(-\beta\Delta - f)$, i.e., $\mathrm{Spec}(-\beta\Delta - f) \subset [0,\infty)$. Let $\{E(\lambda)\}$ be the spectral decomposition of the operator $-\beta\Delta - f$. Then $E(-0) = 0$. Moreover, $E(0) = 0$ holds. Indeed, if one supposes the contrary, then there exists a $\psi \neq 0$, such that

$$\psi \in E(0)L^2(\mathbb{R}^d) \quad \text{and} \quad (-\beta\Delta - f)\psi = 0.$$

Thus, we have $f\psi = -\beta\Delta\psi$, which implies that

$$f\psi \in \mathrm{Ran}(-\beta\Delta) = \mathrm{Dom}(-\beta\Delta)^{-1}$$

and

$$\psi = (-\beta\Delta)^{-1}f\psi.$$

Hence we get $\sqrt{f}\psi = \sqrt{f}(-\beta\Delta)^{-1}f\psi = (\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f})\sqrt{f}\psi$. This contradicts the condition $\|\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}\| < 1$ because $\sqrt{f}\psi \in L^2(\mathbb{R}^d)$ belong to the eigenvalue of 1 of the operator $\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}$. Therefore, we obtain densely defined non-negative self-adjoint operator,

$$(-\beta\Delta - f)^{-1} := \int_0^\infty \frac{dE(\lambda)}{\lambda}.$$

The boundedness of $(-\beta\Delta - f)_\Lambda^{-1}$ follows from the estimates

$$\begin{aligned}\|(-\beta\Delta - f)_\Lambda^{-1}\| &= \sup_{\|\phi\|=1}\int_0^\infty \frac{d\langle\chi_\Lambda\phi, E(\lambda)\chi_\Lambda\phi\rangle}{\lambda}\\ &= \sup_{\|\phi\|=1}\lim_{\delta\downarrow 0}\int_0^\infty \frac{d\langle\chi_\Lambda\phi, E(\lambda)\chi_\Lambda\phi\rangle}{\delta+\lambda} = \sup_{\|\phi\|=1}\lim_{\delta\downarrow 0}\langle\chi_\Lambda\phi, S\chi_\Lambda\phi\rangle\\ &\le \sup_{\|\phi\|=1}\left[\langle\phi,(-\beta\Delta)_\Lambda^{-1}\phi\rangle + \frac{\|\sqrt{f}(\delta-\beta\Delta)^{-1}\chi_\Lambda\phi\|^2}{1-\|\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}\|}\right]\\ &\le \|(-\beta\Delta)_\Lambda^{-1}\| + \frac{\|f\|_\infty\|(\delta-\beta\Delta)_\Lambda^{-1}\|^2}{1-\|\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}\|} < \infty.\end{aligned}$$

To derive Eq. (4.2), we exploit the operator (4.3) for $\delta\downarrow 0$,

$$\begin{aligned}\langle f,(\delta-\beta\Delta-f)^{-1}f\rangle &= \langle\sqrt{f},\sum_{n=1}^{\infty}(\sqrt{f}(\delta-\beta\Delta)^{-1}\sqrt{f})^n\sqrt{f}\rangle = -\langle\sqrt{f},\sqrt{f}\rangle + \langle\sqrt{f},(1-\sqrt{f}(\delta-\beta\Delta)^{-1}\sqrt{f})^{-1}\sqrt{f}\rangle\\ &\to -\int f dx + \langle\sqrt{f},(1-\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f})^{-1}\sqrt{f}\rangle,\end{aligned}$$

where we used the convergence

$$\sqrt{f}(\delta-\beta\Delta)^{-1}\sqrt{f} \to \sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}$$

in the *operator norm*. The latter is a direct consequence of the spectral theorem and the dominated convergence theorem. On the other hand, we notice that for $\delta\downarrow 0$ one gets the following convergence by the monotone convergence theorem:

$$\langle f,(\delta-\beta\Delta-f)^{-1}f\rangle = \int_0^\infty \frac{d\langle\chi_\Lambda f, E(\lambda)\chi_\Lambda f\rangle}{\delta+\lambda} \to \int_0^\infty \frac{d\langle\chi_\Lambda f, E(\lambda)\chi_\Lambda f\rangle}{\lambda} = \langle f,(-\beta\Delta-f)_\Lambda^{-1}f\rangle.$$

Therefore, the equality (4.2) is proven. ■

Theorem 4.8: *For any bounded measurable function $f\ge 0$ with bounded support and for any bounded measurable subset Λ of $\mathbb{R}^d$ satisfying* $\operatorname{supp} f\subset\Lambda$*, we have the following limits:*

$$\begin{aligned}P(t) &:= \lim_{\kappa\to\infty}\frac{1}{\kappa^{d-2}}\log\int_{Q(\mathbb{R}^d)} e^{t\kappa^{-2}\langle f(\cdot/\kappa),\xi\rangle}\nu_\rho(d\xi)\\ &= \begin{cases}\rho_c t\int_{\mathbb{R}^d} f(x)dx + (\rho-\rho_c)t^2\langle f,(-\beta\Delta - tf)_\Lambda^{-1}f\rangle & \text{for } t\in(-\infty,\|\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}\|^{-1})\\ \infty & \text{for } t\in[\|\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}\|^{-1},\infty).\end{cases}\end{aligned}$$

Remark 4.9:

(1) If $t<\|\sqrt{f}(\beta\Delta)_\Lambda^{-1}\sqrt{f}\|^{-1}$, then from Lemma 4.7 we obtain the expression for the function $P(t)$,

$$P(t) = \rho_c t\int_{\mathbb{R}^d} f(x)dx + (\rho-\rho_c)t\langle\sqrt{f},[1-t\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}]^{-1}\sqrt{f}\rangle. \tag{4.4}$$

(2) By Lemma 3.9 and Lemma 3.7 the operator $\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}$ is a non-negative compact opera-

tor. Let $\{\varphi_n\}$ *be a CONS of* $L^2(\mathbb{R}^d)$*, which consists of the eigenfunctions of this operator. We order the corresponding eigenvalues as*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \cdots \geq 0.$$

Then the Perron–Frobenius theorem yields

$$\lambda_1 = \|\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}\| \quad \text{and} \quad \langle \sqrt{f}, \varphi_1 \rangle > 0.$$

By the above remark (1), we obtain

$$P(t) = \rho_c t \int_{\mathbb{R}^d} f(x)dx + (\rho - \rho_c) t \sum_{n=1}^{\infty} \frac{|\langle \sqrt{f}, \varphi_n \rangle|^2}{1 - t\lambda_n} \quad \text{for} \quad t < \lambda_1^{-1},$$

which ensures the (essential) smoothness of P,

$$P \text{ is a } C^\infty \text{ function on} (-\infty, \lambda_1^{-1}) \quad \text{and} \quad \lim_{t \uparrow \lambda_1^{-1}} P(t) = \infty.$$

(3) *Below we prove that the limits of the function P for the components* $\mu_{K,\rho}$ *and* $\mu_K^{(\mathrm{det})}$ *of the boson random point processes have the following forms:*

$$\begin{aligned} P_{K\beta,\rho}(t) &:= \lim_{\kappa\to\infty} \frac{1}{\kappa^{d-2}} \log \int_{Q(\mathbb{R}^d)} e^{t\kappa^{-2}\langle f(\cdot/\kappa), \xi \rangle} \mu_{K\beta,\rho}(d\xi) \\ &= \begin{cases} \rho t \langle \sqrt{f}, [1 - t\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}]^{-1}\sqrt{f} \rangle & \text{for } t \in (-\infty, \|\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}\|^{-1}) \\ \infty & \text{for } t \in [\|\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}\|^{-1}, \infty) \end{cases} \end{aligned}$$

and

$$\begin{aligned} P_{K\beta}^{(\mathrm{det})}(t) &:= \lim_{\kappa\to\infty} \frac{1}{\kappa^{d-2}} \log \int_{Q(\mathbb{R}^d)} e^{t\kappa^{-2}\langle f(\cdot/\kappa), \xi \rangle} \mu_{K\beta}^{(\mathrm{det})}(d\xi) \\ &= \begin{cases} \rho_c t \int_{\mathbb{R}^d} f(x)dx & \text{for } t \in (-\infty, \|\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}\|^{-1}) \\ \infty & \text{for } t \in [\|\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}\|^{-1}, \infty). \end{cases} \end{aligned} \tag{4.5}$$

Proof (of Theorem 4.8): The proof consists of two parts corresponding to $t<0$ and $t>0$. (The case $t=0$ is obvious.)

For $t<0$, it is enough to show that

$$P(-1) = \lim_{\kappa\to\infty} \frac{1}{\kappa^{d-2}} \log \int_{Q(\mathbb{R}^d)} e^{-\kappa^{-2}\langle f(\cdot/\kappa), \xi \rangle} \nu_\rho(d\xi) = -\rho \int_{\mathbb{R}^d} f(x)dx + (\rho - \rho_c)\langle f, (-\beta\Delta + f)_\Lambda^{-1} f \rangle.$$

To this end notice that from (2.2) and (2.3), together with the unitary transformation U_κ, one obtains the representation

$$\begin{aligned} &\frac{1}{\kappa^{d-2}} \log \int_{Q(\mathbb{R}^d)} e^{-\kappa^{-2}\langle f(\cdot/\kappa), \xi \rangle} \nu_\rho(d\xi) \\ &= -\frac{\rho - \rho_c}{\kappa^{d-2}} \langle U_\kappa \sqrt{1 - e^{-\kappa^{-2} f(\cdot/\kappa)}}, U_\kappa (1 + \sqrt{1 - e^{-\kappa^{-2} f(\cdot/\kappa)}} K_{\kappa\Lambda}^\beta \sqrt{1 - e^{-\kappa^{-2} f(\cdot/\kappa)}})^{-1} \sqrt{1 - e^{-\kappa^{-2} f(\cdot/\kappa)}} \rangle \\ &\quad - \frac{1}{\kappa^{d-2}} \log \operatorname{Det}[1 + U_\kappa (1 - e^{-\kappa^{-2} f(\cdot/\kappa)}) K_{\kappa\Lambda}^\beta U_\kappa^{-1}] \\ &= -(\rho - \rho_c) \langle \sqrt{f_\kappa^{(-)}}, (1 + \sqrt{f_\kappa^{(-)}} \kappa^{-2} K_\Lambda^{\beta/\kappa^2} \sqrt{f_\kappa^{(-)}})^{-1} \sqrt{f_\kappa^{(-)}} \rangle \end{aligned}$$

$$-\frac{1}{\kappa^d}\mathrm{Tr}[\sqrt{f_\kappa^{(-)}}K_\Lambda^{\beta/\kappa^2}\sqrt{f_\kappa^{(-)}}]-\frac{1}{\kappa^{d-2}}\log\,\mathrm{Det}_2[1+f_\kappa^{(-)}\kappa^{-2}K_\Lambda^{\beta/\kappa^2}],$$

see Definition 3.4. Then we apply Lemma 3.5 and Lemma 3.7 to the first term, Lemma 3.8 to the second term, and Lemma 3.9 with (4.1) to the third term to obtain

$$P(-1)=-(\rho-\rho_c)\langle\sqrt{f},[1+\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}]^{-1}\sqrt{f}\rangle-\rho_c\int_{\mathbb{R}^d}f(x)dx.$$

Now it is sufficient to check the identity,

$$\langle\sqrt{f},[1+\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}]^{-1}\sqrt{f}\rangle=\int_{\mathbb{R}^d}f(x)dx-\langle f,(-\beta\Delta+f)_\Lambda^{-1}f\rangle. \tag{4.6}$$

Note that the inequality

$$(-\beta\Delta+f)_\Lambda^{-1}\le(-\beta\Delta)_\Lambda^{-1}$$

yields that $(-\beta\Delta+f)_\Lambda^{-1}$ is bounded. Since the operators $(\epsilon-\beta\Delta)^{-1}$, $(\epsilon-\beta\Delta+f)^{-1}$ are bounded and non-negative for any $\epsilon>0$, we get

$$\sqrt{f}(\epsilon-\beta\Delta)^{-1}\sqrt{f}-\sqrt{f}(\epsilon-\beta\Delta+f)^{-1}\sqrt{f}=\sqrt{f}(\epsilon-\beta\Delta)^{-1}\sqrt{f}\sqrt{f}(\epsilon-\beta\Delta+f)^{-1}\sqrt{f}.$$

It gives

$$\sqrt{f}(\epsilon-\beta\Delta)^{-1}\sqrt{f}=(1+\sqrt{f}(\epsilon-\beta\Delta)^{-1}\sqrt{f})\sqrt{f}(\epsilon-\beta\Delta+f)^{-1}\sqrt{f},$$

which implies

$$1-(1+\sqrt{f}(\epsilon-\beta\Delta)^{-1}\sqrt{f})^{-1}=\sqrt{f}(\epsilon-\beta\Delta+f)^{-1}\sqrt{f}.$$

Hence, to verify (4.6), it is enough to prove that

$$\sqrt{f}(\epsilon-\beta\Delta+f)^{-1}\sqrt{f}\to\sqrt{f}(-\beta\Delta+f)_\Lambda^{-1}\sqrt{f}\quad\text{weakly}, \tag{4.7}$$

$$\sqrt{f}(\epsilon-\beta\Delta)^{-1}\sqrt{f}\to\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}\quad\text{in norm}. \tag{4.8}$$

To show (4.7), let $\{E(\lambda)\}$ be the spectral decomposition of $-\beta\Delta+f$. Since

$$\int_0^\infty\frac{d\langle\sqrt{f}\phi,E(\lambda)\sqrt{f}\phi\rangle}{\lambda}=\langle\phi,\sqrt{f}(-\beta\Delta+f)_\Lambda^{-1}\sqrt{f}\phi\rangle\le\langle\phi,\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}\phi\rangle<\infty$$

holds for $\phi\in L^2(\mathbb{R}^d)$, the dominated convergence theorem yields the limit,

$$|\langle\phi,\sqrt{f}(-\beta\Delta+f)_\Lambda^{-1}\sqrt{f}\phi\rangle-\langle\phi,\sqrt{f}(\epsilon-\beta\Delta+f)_\Lambda^{-1}\sqrt{f}\phi\rangle|=\int_0^\infty\left(\frac{1}{\lambda}-\frac{1}{\lambda+\epsilon}\right)d\langle\sqrt{f}\phi,E(\lambda)\sqrt{f}\phi\rangle\to 0.$$

To show (4.8), we use the Fourier transformation. Put

$$\|\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}-\sqrt{f}(\epsilon-\beta\Delta)_\Lambda^{-1}\sqrt{f}\|=\sup_{\|\phi\|=1}\int_{\mathbb{R}^d}\frac{\epsilon|\widetilde{\sqrt{f}\phi}(p)|^2}{\beta|p|^2(\epsilon+\beta|p|^2)}dp=:D.$$

When $d>4$, we obtain that

$$D \le \sup_{\|\phi\|=1} \int_{|p|<1} \frac{\epsilon \|\widetilde{\sqrt{f}\phi}\|_\infty^2}{\beta^2 |p|^4} dp + \sup_{\|\phi\|=1} \int_{|p|\ge 1} \frac{\epsilon |\widetilde{\sqrt{f}\phi}(p)|^2}{\beta^2} dp \le \frac{\epsilon}{\beta^2}(c_d \|f\|_{L^1} + \|f\|_\infty) \to 0$$

for $\epsilon \to 0$. When $2 < d < 4$, we get

$$D \le \sup_{\|\phi\|=1} \int_{\mathbb{R}^d} \frac{\epsilon \|\widetilde{\sqrt{f}\phi}\|_\infty^2}{\beta |p|^2 (\epsilon + \beta |p|^2)} dp \le \frac{\epsilon^{(d-2)/2}}{\beta^{d/2}} \int_{\mathbb{R}^d} \frac{\|f\|_{L^1} d\tilde{p}}{(2\pi)^d |\tilde{p}|^2 (1 + |\tilde{p}|^2)} \to 0,$$

as $\epsilon \to 0$. Here we used the bounds $\|\widetilde{\sqrt{f}\phi}\|_\infty \le (2\pi)^{-d/2} \|f\|_{L^1}^{1/2} \|\phi\|$ and $\|\widetilde{\sqrt{f}\phi}\| \le \|f\|_\infty^{1/2} \|\phi\|$ and changed the integral variable $p = \tilde{p}\sqrt{\epsilon/\beta}$ in the latter integral.

Similarly for $d=4$, we obtain the limit,

$$\begin{aligned} D &\le \sup_{\|\phi\|=1} \int_{|p|<1} \frac{\epsilon \|\widetilde{\sqrt{f}\phi}\|_\infty^2}{\beta |p|^2 (\epsilon + \beta |p|^2)} dp + \sup_{\|\phi\|=1} \int_{|p|\ge 1} \frac{\epsilon |\widetilde{\sqrt{f}\phi}(p)|^2}{\beta^2} dp \\ &\le \frac{\epsilon^{(d-2)/2}}{\beta^{d/2}} \int_{|\tilde{p}|<\sqrt{\beta/\epsilon}} \frac{\|f\|_{L^1} d\tilde{p}}{(2\pi)^d |\tilde{p}|^2 (1 + |\tilde{p}|^2)} + \frac{\epsilon}{\beta^2} \|f\|_\infty \\ &\le c \|f\|_{L^1} \frac{\epsilon}{\beta^2} \log\left(1 + \frac{\beta}{\epsilon}\right) + \frac{\epsilon}{\beta^2} \|f\|_\infty \to 0, \end{aligned}$$

when $\epsilon \to 0$.

For $t > 0$, It is enough to show

$$P(1) = \begin{cases} \rho_c \int_{\mathbb{R}^d} f(x) dx + (\rho - \rho_c) \langle f, (-\beta\Delta - f)_\Lambda^{-1} f \rangle & \text{for } \|\sqrt{f}(-\beta\Delta)_\Lambda^{-1} \sqrt{f}\| < 1 \\ \infty & \text{for } \|\sqrt{f}(-\beta\Delta)_\Lambda^{-1} \sqrt{f}\| \ge 1. \end{cases}$$

When $\|\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}\| < 1$, then by Lemma 3.7 and Lemma 3.3 we have

$$\|\sqrt{f_\kappa^{(+)}} \kappa^{-2} K_\Lambda^{\beta/\kappa^2} \sqrt{f_\kappa^{(+)}}\| = \|\sqrt{e^{\kappa^{-2} f(\cdot/\kappa)} - 1} K_{\kappa\Lambda}^{\beta} \sqrt{e^{\kappa^{-2} f(\cdot/\kappa)} - 1}\| < 1$$

for κ large enough, see Definition 3.4. We also use Lemma 3.3 and Theorem 2.1, parts (2) and (4) to obtain the representation,

$$\begin{aligned} &\frac{1}{\kappa^{d-2}} \log \int_{Q(\mathbb{R}^d)} e^{\kappa^{-2}\langle f(\cdot/\kappa), \xi \rangle} \nu_\rho(d\xi) \\ &= \frac{\rho - \rho_c}{\kappa^{d-2}} \langle U_\kappa \sqrt{e^{\kappa^{-2} f(\cdot/\kappa)} - 1}, U_\kappa (\sqrt{e^{\kappa^{-2} f(\cdot/\kappa)} - 1} K_{\kappa\Lambda}^{\beta} \sqrt{e^{\kappa^{-2} f(\cdot/\kappa)} - 1})^{-1} \sqrt{e^{\kappa^{-2} f(\cdot/\kappa)} - 1} \rangle \\ &- \frac{1}{\kappa^{d-2}} \log \operatorname{Det}[1 - U_\kappa \sqrt{e^{\kappa^{-2} f(\cdot/\kappa)} - 1} K_{\kappa\Lambda}^{\beta} \sqrt{e^{\kappa^{-2} f(\cdot/\kappa)} - 1} U_\kappa^{-1}] \\ &= (\rho - \rho_c) \langle \sqrt{f_\kappa^{(+)}}, (1 - \sqrt{f_\kappa^{(+)}} \kappa^{-2} K_\Lambda^{\beta/\kappa^2} \sqrt{f_\kappa^{(+)}})^{-1} \sqrt{f_\kappa^{(+)}} \rangle \\ &+ \frac{1}{\kappa^d} \operatorname{Tr}[f_\kappa^{(+)} K_\Lambda^{\beta/\kappa^2}] - \frac{1}{\kappa^{d-2}} \log \operatorname{Det}_2[1 - f_\kappa^{(+)} \kappa^{-2} K_\Lambda^{\beta/\kappa^2}]. \end{aligned}$$

Applying Lemma 3.7 and Lemma 3.5 to the first term, Lemma 3.8 and Lemma 3.5 to the second term, and Lemma 3.9 to the third term, we get

$$P(1) = \rho_c \int_{\mathbb{R}^d} f(x)dx + (\rho - \rho_c)\langle \sqrt{f}, [1 - \sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}]^{-1}\sqrt{f}\rangle.$$

Then Lemma 4.7 proves the case $\|\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}\| < 1$.

When $\|\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}\| > 1$, we apply U_κ and Lemma 3.7 and Lemma 3.3 to find that

$$\|\sqrt{e^{f(\cdot/\kappa)/\kappa^2} - 1}K_{\kappa\Lambda}^{\beta}\sqrt{e^{f(\cdot/\kappa)/\kappa^2} - 1}\| \geq 1,$$

for κ large enough. Therefore, we get from Theorem 2.1, parts (2) and (4) that

$$\lim_{\kappa\to\infty} \int_{Q(\mathbb{R}^d)} e^{\langle f(\cdot/\kappa)/\kappa^2, \xi\rangle} \nu_\rho(d\xi) = \infty.$$

When $\|\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}\| = 1$, then applying Lemma 3.7 and transformation U_κ, we find for large κ the estimate

$$\|\sqrt{e^{f(\cdot/\kappa)/\kappa^2} - 1}K_{\kappa\Lambda}^{\beta}\sqrt{e^{f(\cdot/\kappa)/\kappa^2} - 1}\| = \|\sqrt{f_\kappa^{(+)}}\kappa^{-2}K_\Lambda^{\beta/\kappa^2}\sqrt{f_\kappa^{(+)}}\| \geq 1 - c\kappa^{-2}.$$

In fact, it is enough to consider the case where the above quantity is smaller than 1. In this case Lemma 3.5, Lemma 3.7, and Lemma 3.1 yield

$$\begin{aligned}
&|\langle \sqrt{f_\kappa^{(+)}}, (\sqrt{f_\kappa^{(+)}}\kappa^{-2}K_\Lambda^{\beta/\kappa^2}\sqrt{f_\kappa^{(+)}})^n\sqrt{f_\kappa^{(+)}}\rangle - \langle \sqrt{f}, (\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f})^n\sqrt{f}\rangle| \\
&\quad \leq |\langle \sqrt{f_\kappa^{(+)}} - \sqrt{f}, (\sqrt{f_\kappa^{(+)}}\kappa^{-2}K_\Lambda^{\beta/\kappa^2}\sqrt{f_\kappa^{(+)}})^n\sqrt{f_\kappa^{(+)}}\rangle| \\
&\quad + |\langle \sqrt{f}, (\sqrt{f_\kappa^{(+)}}\kappa^{-2}K_\Lambda^{\beta/\kappa^2}\sqrt{f_\kappa^{(+)}})^n(\sqrt{f_\kappa^{(+)}} - \sqrt{f})\rangle| \\
&\quad + |\langle \sqrt{f}, \{(\sqrt{f_\kappa^{(+)}}\kappa^{-2}K_\Lambda^{\beta/\kappa^2}\sqrt{f_\kappa^{(+)}})^n - (\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f})^n\}\sqrt{f}\rangle| \\
&\quad \leq \frac{\|f\|_\infty}{\kappa^2}(1 + e^{\|f\|_\infty/\kappa^2})e^{\|f\|_\infty/\kappa^2}\langle \sqrt{f}, \sqrt{f}\rangle \\
&\quad + n\langle \sqrt{f}, \sqrt{f}\rangle \|\sqrt{f_\kappa^{(+)}}\kappa^{-2}K_\Lambda^{\beta/\kappa^2}\sqrt{f_\kappa^{(+)}} - \sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}\| \\
&\quad \leq c\frac{n+1}{\kappa^2}.
\end{aligned}$$

Together with Theorem 2.1, part (2), this estimate gives the limit

$$\begin{aligned}
&\frac{1}{\kappa^{d-2}}\log \int_{Q(\mathbb{R}^d)} e^{\langle f(\cdot/\kappa)/\kappa^2, \xi\rangle}\mu_{K\beta,(\rho-\rho_c)}(d\xi) \\
&\quad = \frac{\rho - \rho_c}{\kappa^{d-2}}\langle \sqrt{e^{f(\cdot/\kappa)/\kappa^2} - 1}, (1 - \sqrt{e^{f(\cdot/\kappa)/\kappa^2} - 1}K_{\kappa\Lambda}^{\beta}\sqrt{e^{f(\cdot/\kappa)/\kappa^2} - 1})^{-1}\sqrt{e^{f(\cdot/\kappa)/\kappa^2} - 1}\rangle \\
&\quad = (\rho - \rho_c)\sum_{n=0}^{\infty} \langle \sqrt{f_\kappa^{(+)}}, (\sqrt{f_\kappa^{(+)}}\kappa^{-2}K_\Lambda^{\beta/\kappa^2}\sqrt{f_\kappa^{(+)}})^n\sqrt{f_\kappa^{(+)}}\rangle \\
&\quad \geq (\rho - \rho_c)\sum_{n=0}^{\infty} \{\langle \sqrt{f}, (\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f})^n\sqrt{f}\rangle \\
&\quad - |\langle \sqrt{f_\kappa^{(+)}}, (\sqrt{f_\kappa^{(+)}}\kappa^{-2}K_\Lambda^{\beta/\kappa^2}\sqrt{f_\kappa^{(+)}})^n\sqrt{f_\kappa^{(+)}}\rangle - \langle \sqrt{f}, (\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f})^n\sqrt{f}\rangle|\} \vee 0 \\
&\quad \geq (\rho - \rho_c)\sum_{n=0}^{\infty} \left\{ |\langle \varphi, \sqrt{f}\rangle|^2 - c\frac{n+1}{\kappa^2} \right\} \vee 0 \geq (\rho - \rho_c)\frac{|\langle \varphi, \sqrt{f}\rangle|^4\kappa^2}{2c} \to \infty,
\end{aligned}$$

as $\kappa \to \infty$. Here we applied U_κ in the second equality, and then the fact that φ is the eigenfunction of the operator $\sqrt{f}(-\beta\Delta)_\Lambda^{-1}\sqrt{f}$ with the largest eigenvalue of 1. Note that $\langle\sqrt{f},\varphi\rangle>0$. In fact, since the integral kernel of this operator is positive on the set $\{x\in\mathbb{R}^d: f>0\}$, one gets that $\varphi>0$ *a.e.* on $\{x\in\mathbb{R}^d: f>0\}$, cf. Remark 4.9, part (2).

The corresponding estimate for $\mu_{K\beta}^{(\mathrm{det})}$ is straightforward. ■

Recall that the Fenchel–Legendre transformation of the function P has the form

$$I(s) := \sup_{s\in\mathbb{R}}(st - P(t)).$$

We apply the Gärtner–Ellis theorem (see, e.g., Ref. 3) to the random variable $\langle f(\cdot/\kappa)/\kappa^d,\xi\rangle$ and the parameter κ^{d-2} to obtain the following LDP.

Theorem 4.10: *(LDP) The random variable $\langle f(\cdot/\kappa)/\kappa^d,\xi\rangle$ satisfies in the condensation region: $\rho>\rho_c$, the LDP with a rate function I ,*

$$\limsup_{\kappa\to\infty}\frac{1}{\kappa^{d-2}}\log\,\nu_\rho\left[\left\langle\frac{1}{\kappa^d}f(./\kappa),\xi\right\rangle\in F\right]\le -\inf_{s\in F}I(s)\quad\text{for arbitrary closed } F\subset\mathbb{R}$$

and

$$\liminf_{\kappa\to\infty}\frac{1}{\kappa^{d-2}}\log\,\nu_\rho\left[\left\langle\frac{1}{\kappa^d}f(./\kappa),\xi\right\rangle\in G\right]\ge -\inf_{s\in G}I(s)\quad\text{for arbitrary open } G\subset\mathbb{R}.$$

Remark 4.11: Note that contribution of the RPF $\mu_{K\beta}^{(\mathrm{det})}$ to the large deviation rate is in a sense marginal, since it only shifts the argument of the rate function $I(s)$ see (4.4). Taking into account the CLT, we see that the characteristic feature of the limit theorems for the ideal boson gas in the presence of the BEC is defined by the convolution with a nontrivial component $\mu_{K\beta,\rho-\rho_c}$ of the measure $\nu_\rho=\mu_{K\beta}^{(\mathrm{det})}*\mu_{K\beta,\rho-\rho_c}$.

V. CONCLUSION

To compare our results for the case of BEC: $\rho>\rho_c$ with the corresponding results for the case $\rho<\rho_c(\beta)$ (normal phase), we would like to recall here the latter, see Refs. 10, 9, and 15.

Let us put $K_z^\beta := zG^\beta(1-zG^\beta)^{-1}$, with $z=\bar{z}(\beta,\rho)\in(0,1)$, which is a unique solution of the equation

$$\rho = K_z^\beta(x,x) = \int_{\mathbb{R}^d}\frac{dp}{(2\pi)^d}\frac{1}{z^{-1}e^{\beta|p|^2}-1} < \rho_c(\beta) = K_{z=1}^\beta(x,x), \tag{5.1}$$

see (1.1) and (1.3). We also put $\nu_\rho := \mu_{K_{\bar{z}}^\beta}^{(\mathrm{det})}$. Then we have the following.

Theorem 5.1: *(Law of large numbers) For $\kappa\to\infty$, the limit*

$$\frac{1}{\kappa^d}\langle f(\cdot/\kappa),\xi\rangle \to \rho\int_{\mathbb{R}^d}f(x)dx$$

holds in $L^2(Q(\mathbb{R}^d),\nu_\rho)$.

Theorem 5.2: *(CLT) Distributions of the random variables,*

$$Z_\kappa = \frac{1}{\kappa^{d/2}}\frac{\langle f(\cdot/\kappa),\xi\rangle - \kappa^d\rho\int_{\mathbb{R}^d}f(x)dx}{\sqrt{\rho+((K_{\bar{z}}^\beta)^2)(x,x)}\|f\|},$$

converge to the standard normal distribution,

$$\lim_{\kappa\to\infty}\int_{Q(\mathbb{R}^d)}e^{itZ_\kappa}\nu_\rho(d\xi) = e^{-t^2/2}.$$

Theorem 5.3: *(LDP) There exists a convex rate function* $\widetilde{I}:\mathbb{R}\mapsto[0,+\infty]$, *such that*

$$\limsup_{\kappa\to\infty}\frac{1}{\kappa^d}\log\,\nu_\rho\left(\frac{1}{\kappa^d}\langle f(\cdot/\kappa),\xi\rangle\in F\right)\leq-\inf_{s\in F}\widetilde{I}(s)\quad\text{for any closed }F\subset\mathbb{R}$$

and

$$\liminf_{\kappa\to\infty}\frac{1}{\kappa^d}\log\,\nu_\rho\left(\frac{1}{\kappa^d}\langle f(\cdot/\kappa),\xi\rangle\in G\right)\geq-\inf_{s\in G}\widetilde{I}(s)\quad\text{for any open }G\subset\mathbb{R}$$

hold.

We summarize the difference between Theorems 5.1–5.3 and Theorems 1.1–1.3 as follows. Let

$$D_\kappa\coloneqq\frac{1}{\kappa^d}\langle f(\cdot/\kappa),\xi\rangle$$

be the random variable corresponding to *empirical density* of particles localized in a domain of the length scale κ.

For the case of the BEC one gets the following.

(i) The random variable D_κ converges for $\kappa\to\infty$ to its expectation value $m=\rho\int_{\mathbb{R}^d}f(x)dx$ in *mean*.
(ii) The laws of the random variable $\kappa^{(d-2)/2}(D_\kappa-m)$ converge for $\kappa\to\infty$ to normal distribution.
(iii) The random variables D_κ manifest a LDP with the parameter κ^{d-2}.

In the normal phase one obtains that (i) also holds, but (ii) holds for $\kappa^{d/2}(D_\kappa-m)$, instead of $\kappa^{(d-2)/2}(D_\kappa-m)$, and (iii) holds with the parameter κ^d, instead of κ^{d-2}.

The comparison shows that there are differences in particle density fluctuations of the ideal boson gas for the BEC regime and in the normal phase, which reminds the large deviation properties for two-phase classical systems, for example, lattice spin models, see, e.g., Ref. 13. The specificity of the BEC is that it is a *quantum phase transition* with particular properties of the quantum fluctuations.[11,20]

Finally, we would like to mention a special case of the *critical point*: $\rho=\rho_c$. In this case the boson RPF is defined by the measure $\nu_{\rho_c}=\mu^{(\mathrm{det})}_{K^\beta_{\bar{z}=1}}$ and it corresponds to the so-called *critical fluctuations*, which should be considered separately. It is known that the *critical quantum* fluctuation is a subtle matter even for such a simple model as the ideal boson gas, see, e.g., Ref. 1. Therefore, it would be instructive to study the boson RPF at the critical point.

Our analysis yields rather trivial result for the LDP. Namely, Theorem 4.10 holds with ν_{ρ_c} and

$$I(s)=\begin{cases}0 & \text{for }\; s=\rho_c\int_{\mathbb{R}^d}f(x)dx\\ \infty & \qquad\text{otherwise.}\end{cases}$$

This immediately follows from (4.5). It is not very surprising, since at the critical point there is no BEC in the ideal boson gas, i.e., we stay in the one phase regime. On the other hand, for the CLT, neither Theorem 4.5 nor Theorem 5.2 hold at the critical point. For example, $((K^\beta_{\bar{z}=1})^2)(x,x)=\infty$ is not defined for $d\leq 4$.

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